Chapter 1

LINEAR EQUATIONS

1.1 Introduction to linear equations

A *linear equation* in n unknowns $x_1, x_2, \cdots, x_n$ is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

where $a_1, a_2, \ldots, a_n, b$ are given real numbers.

For example, with x and y instead of x_1 and x_2, the linear equation $2x + 3y = 6$ describes the line passing through the points (3, 0) and (0, 2).

Similarly, with x, y and z instead of x_1, x_2 and x_3, the linear equation $2x + 3y + 4z = 12$ describes the plane passing through the points (6, 0, 0), (0, 4, 0), (0, 0, 3).

A *system* of m linear equations in n unknowns $x_1, x_2, \cdots, x_n$ is a family of linear equations

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m.
\end{aligned}$$

We wish to determine if such a system has a solution, that is to find out if there exist numbers $x_1, x_2, \cdots, x_n$ which satisfy each of the equations simultaneously. We say that the system is *consistent* if it has a solution. Otherwise the system is called *inconsistent*.

Note that the above system can be written concisely as

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, \quad i = 1,\, 2, \cdots, m.$$

The matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is called the *coefficient matrix* of the system, while the matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

is called the *augmented matrix* of the system.

Geometrically, solving a system of linear equations in two (or three) unknowns is equivalent to determining whether or not a family of lines (or planes) has a common point of intersection.

EXAMPLE 1.1.1 Solve the equation

$$2x + 3y = 6.$$

Solution. The equation $2x + 3y = 6$ is equivalent to $2x = 6 - 3y$ or $x = 3 - \frac{3}{2}y$, where y is arbitrary. So there are infinitely many solutions.

EXAMPLE 1.1.2 Solve the system

$$\begin{aligned} x + y + z &= 1 \\ x - y + z &= 0. \end{aligned}$$

Solution. We subtract the second equation from the first, to get $2y = 1$ and $y = \frac{1}{2}$. Then $x = y - z = \frac{1}{2} - z$, where z is arbitrary. Again there are infinitely many solutions.

EXAMPLE 1.1.3 Find a polynomial of the form $y = a_0 + a_1x + a_2x^2 + a_3x^3$ which passes through the points $(-3, -2)$, $(-1, 2)$, $(1, 5)$, $(2, 1)$.