Chapter One

Complex Numbers

1.1 Introduction. Let us hark back to the first grade when the only numbers you knew were the ordinary everyday integers. You had no trouble solving problems in which you were, for instance, asked to find a number x such that $3x = 6$. You were quick to answer ”2”. Then, in the second grade, Miss Holt asked you to find a number x such that $3x = 8$. You were stumped—there was no such ”number”! You perhaps explained to Miss Holt that $3(2) = 6$ and $3(3) = 9$, and since 8 is between 6 and 9, you would somehow need a number between 2 and 3, but there isn’t any such number. Thus were you introduced to ”fractions.”

These fractions, or rational numbers, were defined by Miss Holt to be ordered pairs of integers—thus, for instance, $(8,3)$ is a rational number. Two rational numbers (n,m) and (p,q) were defined to be equal whenever $nq = pm$. (More precisely, in other words, a rational number is an equivalence class of ordered pairs, *etc.*) Recall that the arithmetic of these pairs was then introduced: the sum of (n,m) and (p,q) was defined by

$$(n,m) + (p,q) = (nq + pm, mq),$$

and the product by

$$(n,m)(p,q) = (np, mq).$$

Subtraction and division were defined, as usual, simply as the inverses of the two operations.

In the second grade, you probably felt at first like you had thrown away the familiar integers and were starting over. But no. You noticed that $(n,1) + (p,1) = (n+p,1)$ and also $(n,1)(p,1) = (np,1)$. Thus the set of all rational numbers whose second coordinate is one behave just like the integers. If we simply abbreviate the rational number $(n,1)$ by n, there is absolutely no danger of confusion: $2 + 3 = 5$ stands for $(2,1) + (3,1) = (5,1)$. The equation $3x = 8$ that started this all may then be interpreted as shorthand for the equation $(3,1)(u,v) = (8,1)$, and one easily verifies that $x = (u,v) = (8,3)$ is a solution. Now, if someone runs at you in the night and hands you a note with 5 written on it, you do not know whether this is simply the integer 5 or whether it is shorthand for the rational number $(5,1)$. What we see is that it really doesn’t matter. What we have ”really” done is expanded the collection of integers to the collection of rational numbers. In other words, we can think of the set of all rational numbers as including the integers–they are simply the rationals with second coordinate 1.

One last observation about rational numbers. It is, as everyone must know, traditional to write the ordered pair (n,m) as $\frac{n}{m}$. Thus n stands simply for the rational number $\frac{n}{1}$, *etc.*

Now why have we spent this time on something everyone learned in the second grade? Because this is almost a paradigm for what we do in constructing or defining the so-called complex numbers. Watch.

Euclid showed us there is no rational solution to the equation $x^2 = 2$. We were thus led to defining even more new numbers, the so-called real numbers, which, of course, include the rationals. This is hard, and you likely did not see it done in elementary school, but we shall assume you know all about it and move along to the equation $x^2 = -1$. Now we define **complex numbers**. These are simply ordered pairs (x,y) of real numbers, just as the rationals are ordered pairs of integers. Two complex numbers are equal only when there are actually the same–that is $(x,y) = (u,v)$ precisely when $x = u$ and $y = v$. We define the sum and product of two complex numbers:

$$(x,y) + (u,v) = (x+u, y+v)$$

and

$$(x,y)(u,v) = (xu - yv, xv + yu)$$

As always, subtraction and division are the inverses of these operations.

Now let's consider the arithmetic of the complex numbers with second coordinate 0:

$$(x,0) + (u,0) = (x+u, 0),$$

and

$$(x,0)(u,0) = (xu, 0).$$

Note that what happens is completely analogous to what happens with rationals with second coordinate 1. We simply use x as an abbreviation for $(x,0)$ and there is no danger of confusion: $x+u$ is short-hand for $(x,0)+(u,0) = (x+u,0)$ and xu is short-hand for $(x,0)(u,0)$. We see that our new complex numbers include a copy of the real numbers, just as the rational numbers include a copy of the integers.

Next, notice that $x(u,v) = (u,v)x = (x,0)(u,v) = (xu,xv)$. Now then, any complex number $z = (x,y)$ may be written

$$
\begin{aligned}
z &= (x,y) = (x,0) + (0,y) \\
&= x + y(0,1)
\end{aligned}
$$

When we let $\alpha = (0,1)$, then we have

$$z = (x,y) = x + \alpha y$$

Now, suppose $z = (x,y) = x + \alpha y$ and $w = (u,v) = u + \alpha v$. Then we have

$$
\begin{aligned}
zw &= (x + \alpha y)(u + \alpha v) \\
&= xu + \alpha(xv + yu) + \alpha^2 yv
\end{aligned}
$$

We need only see what α^2 is: $\alpha^2 = (0,1)(0,1) = (-1,0)$, and we have agreed that we can safely abbreviate $(-1,0)$ as -1. Thus, $\alpha^2 = -1$, and so

$$zw = (xu - yv) + \alpha(xv + yu)$$

and we have reduced the fairly complicated definition of complex arithmetic simply to ordinary real arithmetic together with the fact that $\alpha^2 = -1$.

Let's take a look at division–the inverse of multiplication. Thus $\frac{z}{w}$ stands for that complex number you must multiply w by in order to get z. An example:

$$
\begin{aligned}
\frac{z}{w} &= \frac{x + \alpha y}{u + \alpha v} = \frac{x + \alpha y}{u + \alpha v} \cdot \frac{u - \alpha v}{u - \alpha v} \\
&= \frac{(xu + yv) + \alpha(yu - xv)}{u^2 + v^2} \\
&= \frac{xu + yv}{u^2 + v^2} + \alpha \frac{yu - xv}{u^2 + v^2}
\end{aligned}
$$

Note this is just fine except when $u^2 + v^2 = 0$; that is, when $u = v = 0$. We may thus divide by any complex number except $0 = (0,0)$.

One final note in all this. Almost everyone in the world except an electrical engineer uses the letter i to denote the complex number we have called α. We shall accordingly use i rather than α to stand for the number $(0,1)$.

Exercises

1. Find the following complex numbers in the form $x + iy$:

a) $(4 - 7i)(-2 + 3i)$ b) $(1 - i)^3$

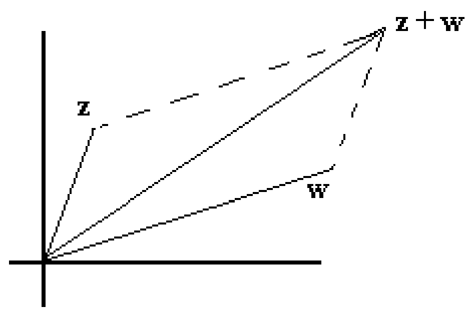
b) $\frac{(5+2i)}{(1+i)}$ c) $\frac{1}{i}$

2. Find all complex $z = (x, y)$ such that

$$z^2 + z + 1 = 0$$

3. Prove that if $wz = 0$, then $w = 0$ or $z = 0$.

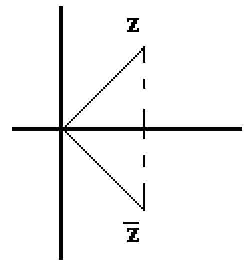
1.2. Geometry. We now have this collection of all ordered pairs of real numbers, and so there is an uncontrollable urge to plot them on the usual coordinate axes. We see at once then there is a one-to-one correspondence between the complex numbers and the points in the plane. In the usual way, we can think of the sum of two complex numbers, the point in the plane corresponding to $z + w$ is the diagonal of the parallelogram having z and w as sides:



We shall postpone until the next section the geometric interpretation of the product of two complex numbers.

The **modulus** of a complex number $z = x + iy$ is defined to be the nonnegative real number $\sqrt{x^2 + y^2}$, which is, of course, the length of the vector interpretation of z. This modulus is traditionally denoted $|z|$, and is sometimes called the **length** of z. Note that $|(x, 0)| = \sqrt{x^2} = |x|$, and so $|\cdot|$ is an excellent choice of notation for the modulus.

The **conjugate** $\overline{z}$ of a complex number $z = x + iy$ is defined by $\overline{z} = x - iy$. Thus $|z|^2 = z\overline{z}$. Geometrically, the conjugate of z is simply the reflection of z in the horizontal axis:



Observe that if $z = x + iy$ and $w = u + iv$, then

$$\begin{aligned} \overline{(z+w)} &= (x+u) - i(y+v) \\ &= (x-iy) + (u-iv) \\ &= \overline{z} + \overline{w}. \end{aligned}$$

In other words, the conjugate of the sum is the sum of the conjugates. It is also true that $\overline{zw} = \overline{z}\,\overline{w}$. If $z = x + iy$, then x is called the **real part** of z, and y is called the **imaginary part** of z. These are usually denoted $\mathrm{Re}\,z$ and $\mathrm{Im}\,z$, respectively. Observe then that $z + \overline{z} = 2\,\mathrm{Re}\,z$ and $z - \overline{z} = 2\,\mathrm{Im}\,z$.

Now, for any two complex numbers z and w consider

$$\begin{aligned} |z+w|^2 &= (z+w)\overline{(z+w)} = (z+w)(\overline{z} + \overline{w}) \\ &= z\overline{z} + (w\overline{z} + \overline{w}z) + w\overline{w} \\ &= |z|^2 + 2\,\mathrm{Re}(w\overline{z}) + |w|^2 \\ &\leq |z|^2 + 2|z||w| + |w|^2 = (|z| + |w|)^2 \end{aligned}$$

In other words,

$$|z+w| \leq |z| + |w|$$

the so-called **triangle inequality**. (This inequality is an obvious geometric fact–can you guess why it is called the *triangle inequality*?)

Exercises

4. a)Prove that for any two complex numbers, $\overline{zw} = \overline{z}\,\overline{w}$.
b)Prove that $\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}$.
c)Prove that $||z| - |w|| \leq |z - w|$.

5. Prove that $|zw| = |z||w|$ and that $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$.

6. Sketch the set of points satisfying

a) $|z - 2 + 3i| = 2$

b) $|z + 2i| \le 1$

c) $\text{Re}(\overline{z} + i) = 4$

d) $|z - 1 + 2i| = |z + 3 + i|$

e) $|z + 1| + |z - 1| = 4$

f) $|z + 1| - |z - 1| = 4$

1.3. Polar coordinates. Now let's look at polar coordinates (r,θ) of complex numbers. Then we may write $z = r(\cos\theta + i\sin\theta)$. In complex analysis, we do not allow r to be negative; thus r is simply the modulus of z. The number θ is called an **argument** of z, and there are, of course, many different possibilities for θ. Thus a complex numbers has an infinite number of arguments, any two of which differ by an integral multiple of 2π. We usually write $\theta = \arg z$. The **principal argument** of z is the unique argument that lies on the interval $(-\pi, \pi]$.

Example. For $1 - i$, we have

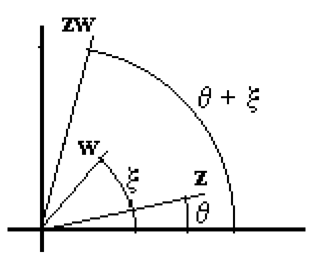
$$\begin{aligned} 1 - i &= \sqrt{2}\,(\cos\left(\tfrac{7\pi}{4}\right) + i\sin\left(\tfrac{7\pi}{4}\right)) \\ &= \sqrt{2}\,(\cos\left(-\tfrac{\pi}{4}\right) + i\sin\left(-\tfrac{\pi}{4}\right)) \\ &= \sqrt{2}\,(\cos\left(\tfrac{399\pi}{4}\right) + i\sin\left(\tfrac{399\pi}{4}\right)) \end{aligned}$$

etc., etc., etc. Each of the numbers $\frac{7\pi}{4}$, $-\frac{\pi}{4}$, and $\frac{399\pi}{4}$ is an argument of $1 - i$, but the principal argument is $-\frac{\pi}{4}$.

Suppose $z = r(\cos\theta + i\sin\theta)$ and $w = s(\cos\xi + i\sin\xi)$. Then

$$\begin{aligned} zw &= r(\cos\theta + i\sin\theta)s(\cos\xi + i\sin\xi) \\ &= rs[(\cos\theta\cos\xi - \sin\theta\sin\xi) + i(\sin\theta\cos\xi + \sin\xi\cos\theta)] \\ &= rs(\cos(\theta + \xi) + i\sin(\theta + \xi)) \end{aligned}$$

We have the nice result that the product of two complex numbers is the complex number whose modulus is the product of the moduli of the two factors and an argument is the sum of arguments of the factors. A picture:



We now define $\exp(i\theta)$, or $e^{i\theta}$ by

$$e^{i\theta} = \cos\theta + i\sin\theta$$

We shall see later as the drama of the term unfolds that this very suggestive notation is an excellent choice. Now, we have in polar form

$$z = re^{i\theta},$$

where $r = |z|$ and θ is any argument of z. Observe we have just shown that

$$e^{i\theta}e^{i\xi} = e^{i(\theta+\xi)}.$$

It follows from this that $e^{i\theta}e^{-i\theta} = 1$. Thus

$$\frac{1}{e^{i\theta}} = e^{-i\theta}$$

It is easy to see that

$$\frac{z}{w} = \frac{re^{i\theta}}{se^{i\xi}} = \frac{r}{s}(\cos(\theta - \xi) + i\sin(\theta - \xi))$$

Exercises

7. Write in polar form $re^{i\theta}$:

a) i b) $1 + i$

c) -2 d) $-3i$

e) $\sqrt{3} + 3i$

8. Write in rectangular form—no decimal approximations, no trig functions:

a) $2e^{i3\pi}$ b) $e^{i100\pi}$

c) $10e^{i\pi/6}$ d) $\sqrt{2}\,e^{i5\pi/4}$

9. a) Find a polar form of $(1 + i)(1 + i\sqrt{3})$.

b) Use the result of a) to find $\cos\left(\frac{7\pi}{12}\right)$ and $\sin\left(\frac{7\pi}{12}\right)$.

10. Find the rectangular form of $(-1 + i)^{100}$.

11. Find all z such that $z^3 = 1$. (Again, rectangular form, no trig functions.)

12. Find all z such that $z^4 = 16i$. (Rectangular form, *etc*.)