PREFACE

In 1932, the author published *Differential equations from the algebraic standpoint*,[1] a book dealing with differential polynomials and algebraic differential manifolds. In the sixteen years which have passed, the work of a number of mathematicians has given fresh substance and new color to the subject. The complete edition of the book having been exhausted, it has seemed proper to prepare a new exposition.

The title *Differential algebra* was suggested by Dr. Kolchin. The body of algebra deals with the operations of addition and multiplication. We are concerned here with three operations—addition, multiplication and differentiation.

If I am not mistaken, the general nature of the subject here treated is now well enough known among mathematicians to permit me to dispense with a detailed introduction, such as was given in A. D. E. My principal task is to show how much the present book owes to my associates. I am referring to H. W. Raudenbush, W. C. Strodt, E. R. Kolchin, Howard Levi, Eli Gourin and Richard M. Cohn.

Cohn's constructive proof of the theorem of zeros will be found in Chapter V. The theorem on embedded manifolds due to Gourin is contained in Chapter II. Chapter VI contains a discussion of Strodt's work on sequences of manifolds.

In Chapters I, III and IX, there are presented portions of Levi's work on ideals of differential polynomials and on the low power theorem. Of Kolchin's investigation of exponents of differential ideals, I have been able to give only a bare idea. Other work of Kolchin, for instance, proofs for the abstract case of results previously established for the analytic case, is given in Chapter II. His work on the Picard–Vessiot theory, which employs the methods of differential algebra, has just appeared in the Annals of Mathematics,[2] and may be permitted to speak for itself.

The contributions of Raudenbush can only be described as fundamental. The basis theorem of Chapter I was, in the analytic case, implicitly contained in A. D. E. It exists there in two parts; the first, the theorem on the completeness of infinite systems; the second, the theorem of zeros. Only casually had I noticed that the two theorems amounted to a basis theorem. I was acquainted with the fact that the theorem on the decomposition of manifolds amounted, in virtue of the theorem of zeros, to a theory of perfect and prime ideals of differential polynomials. In the summer of 1933, I suggested to Raudenbush the problem of constructing a theory of perfect ideals which would be valid in the abstract case. This he accomplished, and, in the course of his work, he brought the basis theorem to its present complete and abstract form. In the proof of

[1] These Colloquium publications, vol. 14. Called below A. D. E.

[2] Kolchin, 14. (See Bibliography, p. 180.)

the basis theorem, the procedure of taking powers is due to Raudenbush. The chains, characteristic sets and methods of reduction existed in the older theorem of completeness.

Raudenbush introduced generic zeros of prime ideals. Here he adapted a method of van der Waerden, which can be traced back to König. Raudenbush gave the first example of a system of differential polynomials with a weak basis. Systems with no strong bases were later produced by Kolchin.

The problems which this book treats are very concrete problems. They deal with situations of the classical theory of differential equations. Seldom would much be lost, as far as the results are concerned, if one limited oneself to the material of classical analysis. The abstract method which we generally employ has, however, a definite utility. It serves to separate algebraic methods from analytic methods. On the whole, it contributes to simplicity, although at times an abstract treatment is less natural than an analytical one. The form in which the results of differential algebra are being presented has thus been deeply influenced by the teachings of Emmy Noether, a prime mover of our period, who, in continuing Julius König's development of Kronecker's ideas, brought mathematicians to know algebra as it was never known before.

In this connection, I should like to say something concerning basis theorems. The basis theorem of Chapter I will be seen to play, in the present theory, the role held by Hilbert's theorem in the theories of polynomial ideals and of algebraic manifolds. When I began to work on algebraic differential equations, early in 1930, van der Waerden's excellent *Moderne Algebra* had not yet appeared. However, Emmy Noether's work of the twenties was available, and there was nothing to prevent one from learning in her papers the value of basis theorems in decomposition problems. Actually, I became acquainted with the basis theorem principle in the writings of Jules Drach[3] on logical integration, writings which date back to 1898. How a basis theorem is employed by him will now be described.

There are two distinct methods for characterizing an irreducible algebraic equation. On the one hand, an equation $f(x) = 0$ is irreducible if $f(x)$ cannot be factored. On the other, there is irreducibility if every equation which is satisfied by a single solution of $f(x) = 0$ is satisfied by all such solutions. The first formulation of irreducibility leads to the notion of irreducible algebraic manifold and to that of irreducible algebraic differential manifold. The second leads to the concept of irreducible system of algebraic differential equations which was employed by Koenigsberger and by Drach. A system of such equations, ordinary or partial, is irreducible if every differential equation which admits a single solution of the system admits all solutions. Drach undertakes to show that, given a system of partial differential equations, the repeated adjunction of new equations will eventually produce an irreducible system. For this he invokes a theorem of Tresse,[4] which states that, in every infinite system

[3] Drach, 4, pp. 292–296.

[4] Acta Mathematica, vol. 18 (1894), p. 4.

of partial differential equations, there is a finite subsystem from which the infinite system can be derived by differentiations and eliminations. A study of Tresse's paper will quickly convince one that he claims for his work a generality which it does not have. The statement of his theorem, and his argument, have a definite meaning only for linear systems.

It has not been possible for me to present all of the material which has been developed since the publication of A. D. E. Thus, I have had to pass by most of Kolchin's study of exponents and a good deal of Levi's work on ideals. Of Strodt's paper, only a sketch is given. My own work on general solutions of equations of the second order in one unknown, and of equations of the first order in two unknowns, is also omitted.

I have tried to give, to the present book, the elementary quality which is possessed by A. D. E. Essentially, no previous knowledge of abstract algebra is necessary. As in A. D. E., a treatment is given of Riquier's existence theorem for orthonomic systems of partial differential equations.

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