Preface to the English edition by V.I. Arnold

Abel's Theorem, claiming that there exists no finite combinations of radicals and rational functions solving the generic algebraic equation of degree 5 (or higher than 5), is one of the first and the most important impossibility results in mathematics.

I had given to Moscow High School children in 1963–1964 a (half year long) course of lectures, containing the topological proof of the Abel theorem.

Starting from the definition of complex numbers and from geometry, the students were led to Riemannian surfaces in a sequence of elementary problems. Next came the basic topological notions, such as the fundamental group, coverings, ramified coverings, their monodromies, braids, etc..

These geometrical and topological studies implied such elementary general notions as the transformations groups and group homomorphisms, kernels, exact sequences, and relativistic ideas. The normal subgroups appeared as those subgroups which are relativistically invariant, that is, do not depend on the choice of the coordinate frame, represented in this case as a numbering or labelling of the group elements.

The regular polyhedra symmetry groups, seen from this point of view, had led the pupils to the five Kepler's cubes, inscribed into the dodecahedron. The 12 edges of each of these cubes are the diagonals of the 12 faces of the dodecahedron.

Kepler had invented these cubes in his *Harmonia Mundi* to describe the distances of the planets from the Sun. I had used them to obtain the natural isomorphism between the dodecahedron rotations group and the group of the 60 even permutations of 5 elements (being the Kepler cubes). This elementary theory of regular polyhedra provides the non-solubility proof of the 5 elements permutation group: it can not be constructed from the commutative groups by a finite sequence of the extensions with commutative kernels.

The situation is quite different for the permutation groups of less than five elements, which are soluble (and responsible for the solvability of the equations of degree smaller than 5). This solubility depends on the inscription of two tetrahedra inside the cube (similar to the inscription of the 5 Kepler cubes inside the dodecahedron and mentioned also by Kepler).

The absence of the non-trivial relativistically invariant symmetry subgroups of the group of rotations of the dodecahedron is an easy result of elementary geometry. Combining these High School geometry arguments with the preceding topological study of the monodromies of the ramified coverings, one immediately obtains the Abel Theorem topological proof, the monodromy group of any finite combination of the radicals being soluble, since the radical monodromy is a cyclical commutative group, whilst the monodromy of the algebraic function $x(a)$ defined by the quintic equation $x^5 + ax + 1 = 0$ is the non-soluble group of the 120 permutations of the 5 roots.

This theory provides more than the Abel Theorem. It shows that the insolvability argument is topological. Namely, no function having the same topological branching type as $x(a)$ is representable as a finite combination of the rational functions and of the radicals.

I hope that my topological proof of this generalized Abel Theorem opens the way to many topological insolvability results. For instance, one should prove the impossibility of representing the generic abelian integrals of genus higher than zero as functions topologically equivalent to the elementary functions.

I attributed to Abel the statements that neither the generic elliptic integrals nor the generic elliptic functions (which are inverse functions of these integrals) are topologically equivalent to any elementary function.

I thought that Abel was already aware of these topological results and that their absence in the published papers was, rather, owed to the underestimation of his great works by the Paris Academy of Sciences (where his manuscript had been either lost or hidden by Cauchy).

My 1964 lectures had been published in 1976 by one of the pupils of High School audience, V.B. Alekseev. He has somewhere algebraized my geometrical lectures.

Some of the topological ideas of my course had been developed by A.G. Khovanskii, who had thus proved some new results on the insolvability of the differential equations. Unfortunately, the topological insolvability proofs are still missing in his theory (as well as in the Poincaré theory of the absence of the holomorphic first integral and in many other insolvability problems of differential equations theory).

I hope that the description of these ideas in the present translation of Alekseev's book will help the English reading audience to participate in the development of this new topological insolvability theory, started with the topological proof of the Abel Theorem and involving, say, the topologically non elementary nature of the abelian integrals as well as the topological non-equivalence to the integrals combinations of the complicated differential equations solutions.

The combinatory study of the Kepler cubes, used in the Abel theorem's proof, is also the starting point of the development of the theory of finite groups. For instance, the five Kepler cubes depend on the 5 Hamilton subgroups of the projective version $\mathrm{PSL}_2(\mathbb{Z}_5)$ of the group of matrices of order 2 whose elements are residues modulo 5.

A Hamilton subgroup consists of 8 elements and is isomorphic to the group $\{\pm 1, \pm i, \pm j, \pm k\}$ of the quaternionics units.

The peculiar geometry of the finite groups includes their squaring monads, which are the oriented graphs whose vertices are the group elements and whose edges connect every element directly to its square.

The $\mathrm{PSL}_2(\mathbb{Z}_5)$ monads theory leads to the unexpected Riemannian surfaces (including the monads as subgraphs), relating Kepler's cubes to the peculiarities of the geometry of elliptic curves.

The $\mathrm{PSL}_2(\mathbb{Z}_7)$ extension of the Hamilton subgroups and of Kepler's cubes leads to the extended four colour problem (for the genus one toroidal surface of an elliptic curve), the 14 Hamilton subgroups providing the proof of the 7 colours necessity for the regular colouring of maps of a toroidal surface).

I hope that these recent theories will be developed further by the readers of this book.

V. Arnold