Chapter 1

Introduction

The interaction between numerical linear algebra and control theory has crucially influenced the development of numerical algorithms for linear systems in the past. Since the performance of a control system can often be measured in terms of eigenvalues or singular values, matrix eigenvalue methods have become an important tool for the implementation of control algorithms. Standard numerical methods for eigenvalue or singular value computations are based on the QR-algorithm. However, there are a number of computational problems in control and signal processing that are not amenable to standard numerical theory or cannot be easily solved using current numerical software packages. Various examples can be found in the digital filter design area. For instance, the task of finding sensitivity optimal realizations for finite word length implementations requires the solution of highly nonlinear optimization problems for which no standard numerical solution algorithms exist.

There is thus the need for a new approach to the design of numerical algorithms that is flexible enough to be applicable to a wide range of computational problems as well as has the potential of leading to efficient and reliable solution methods. In fact, various tasks in linear algebra and system theory can be treated in a unified way as optimization problems of smooth functions on Lie groups and homogeneous spaces. In this way the powerful tools of differential geometry and Lie group theory become available to study such problems.

Higher order local convergence properties of iterative matrix algorithms are in many instances proven by means of tricky estimates. E.g., the Jacobi method, essentially, is an optimization procedure. The idea behind the proof of local quadratic convergence for the cyclic Jacobi method applied to a Hermitian matrix lies in the fact that one can estimate the amount of descent per sweep, see Henrici (1958) [Hen58]. Later on, by several authors these ideas where transferred to similar problems and even refined, e.g., Jacobi for the symmetric eigenvalue problem, Kogbetliantz (Jacobi) for SVD, skew-symmetric Jacobi, etc..

The situation seems to be similar for QR-type algorithms. Looking first at Rayleigh quotient iteration, neither Ostrowski (1958/59) [Ost59] nor Parlett [Par74] use Calculus to prove local cubic convergence.

About ten years ago there appeared a series of papers where the authors studied the global convergence properties of QR and RQI by means of dynamical systems methods, see Batterson and Smillie [BS89a, BS89b, BS90], Batterson [Bat95], and Shub and Vasquez [SV87]. To our knowledge these papers where the only ones where Global Analysis was applied to QR-type algorithms.

From our point of view there is a lack in studying the local convergence properties of matrix algorithms in a systematic way. The methodologies for different algorithms are often also different. Moreover, the possibility of considering a matrix algorithm atleast locally as a discrete dynamical system on a homogenous space is often overseen. In this thesis we will take this point of view. We are able to (re)prove higher order convergence for several wellknown algorithms and present some efficient new ones.

This thesis contains three parts.

At first we present a Calculus approach to the local convergence analysis of the Jacobi algorithm. Considering these algorithms as selfmaps on a manifold (i.e., projective space, isospectral or flag manifold, etc.) it turns out, that under the usual assumptions on the spectrum, they are differentiable maps around certain fixed points. For a wide class of Jacobi-type algorithms this is true due to an application of the Implicit Function Theorem, see [HH97, HH00, Hüp96, HH95, HHM96]. We then generalize the Jacobi approach to socalled Block Jacobi methods. Essentially, these methods are the manifold version of the socalled grouped variable approach to coordinate descent, wellknown to the optimization community.

In the second chapter we study the nonsymmetric eigenvalue problem introducing a new algorithm for which we can prove quadratic convergence. These methods are based on the idea to solve lowdimensional Sylvester equations again and again for improving estimates of invariant subspaces.

At third, we will present a new shifted QR-type algorithm, which is somehow the *true* generalization of the Rayleigh Quotien Iteration (RQI) to a full symmetric matrix, in the sense, that not only one column (row) of the matrix converges cubically in norm, but the off-diagonal part as a whole. Rather than being a scalar, our shift is matrix valued. A prerequisite for studying this algorithm, called Parallel RQI, is a detailed local analysis of the classical RQI itself. In addition, at the end of that chapter we discuss the local convergence properties of the shifted QR-algorithm. Our main result for this topic is that there cannot exist a smooth shift strategy ensuring quadratic convergence.

In this thesis we do not answer questions on global convergence. The algorithms which are presented here are all locally smooth self mappings of manifolds with vanishing first derivative at a fixed point. A standard argument using the mean value theorem then ensures that there exists an open neighborhood of that fixed point which is invariant under the iteration of the algorithm. Applying then the contraction theorem on the closed neighborhood ensures convergence to that fixed point and moreover that the fixed point is isolated. Most of the algorithms turn out to be discontinous far away from their fixed points but we will not go into this.

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