"Al-gebra and Co-gebra
are brother and sister"

Zbigniew Oziewicz

Seht Ihr den Mond dort stehen
er ist nur halb zu sehen
und ist doch rund und schön
so sind gar manche Sachen
die wir getrost belachen
weil unsre Augen sie nicht sehn.

Matthias Claudius

Preface

This 'Habilitationsschrift' is the second incarnation of itself – and still in a *status nascendi*. The original text was planned to contain Clifford *algebras* of an arbitrary bilinear form, now called Quantum Clifford Algebras (QCA) and their beautiful application to quantum field theory (QFT). However, while proceeding this way, a major change in paradigm took place after the 5th Clifford conference held in Ixtapa 1999. As a consequence the first incarnation of this work faded away without reaching a properly typeset form, already in late 2000.

What had happened? During the 5th Clifford conference at Ixtapa a special session dedicated to Gian-Carlo Rota, who was assumed to attend the conference but died in Spring 1999, took place. Among other impressive retrospectives delivered during this occasion about Rota and his work, Zbigniew Oziewicz explained the Rota-Stein cliffordization process and coined the term 'Rota-sausage' for the corresponding tangle – for obvious reason as you will see in the main text. This approach to the Clifford product turned out to be superior to all other previously achieved approaches in elegance, efficiency, naturalness and beauty – for a discussion of 'beautiness' in mathematics, see [116], Chap. X, '*The Phenomenology of Mathematical Beauty*'. So I had decided to revise the whole writing. During 2000, beside being very busy with editing [4], it turned out, that not only a rewriting was necessary, but that taking a new starting point changes the whole tale!

A major help in entering the Hopf gebra business for Graßmann and Clifford algebras and cliffordization was the CLIFFORD package [2] developed by Rafał Abłamowicz. During a collaboration with him which took place in Konstanz in Summer 1999, major problems had been solved which led to the formation of the BIGEBRA package [3] in December 1999. The package proved to be calculationable stable and useful for the first time in Autumn 2000 during a joint work with Zbigniew Oziewicz, where many involved computations were successfully performed. The requirements of this lengthy computations completed the BIGEBRA package more or less. Its final form was produced jointly with Rafał Abłamowicz in Cookeville, September 2001.

The possibility of automated calculations and the knowledge of functional quantum field theory [128, 17] allowed to produce a first important result. The relation between time- and normal-ordered operator products and correlation functions was revealed to be a special kind of cliffordization which introduces an antisymmetric (symmetric for bosons) part in the bilinear form of the Clifford product [56]. For short, QCAs deal with time-ordered monomials while regular Clifford algebras of a symmetric bilinear form deal with normal-ordered monomials.

It seemed to be an easy task to translate with benefits all of the work described in [129, 48, 60, 50, 54, 55] into the hopfish framework. But examining Ref. [55] it showed up that the standard literature on Hopf algebras is set up in a too narrow manner so that some concepts had to be generalized first.

Much worse, Oziewicz showed that given an invertible scalar product B the Clifford bi-convolution $\mathcal{C}\ell(B, B^{-1})$, where the Clifford co-product depends on the co-scalar product B^{-1}, has *no antipode* and is therefore not a Hopf algebra at all. But the antipode played *the central role* in Connes-Kreimer renormalization theory [82, 33, 34, 35]. Furthermore the topological meaning and the group-like structure are tied to Hopf algebras, not to convolution semigroups. This motivated Oziewicz to introduce a second *independent* bilinear form, the co-scalar product C in the Clifford bi-convolution $\mathcal{C}\ell(B, C)$, $C \neq B^{-1}$ which is antipodal and therefore Hopf. A different solution was obtained jointly in [59].

Meanwhile QCAs made their way into differential geometry and showed up to be useful in Einstein-Cartan-Kähler theory with teleparallel connections developed by J. Vargas, see [131] and references therein. It was clear for some time that also differential forms, the Cauchy-Riemann differential equations and cohomology have to be revisited in this formalism. This belongs not to our main theme and will be published elsewhere [58].

Another source supplied ideas – geometry and robotics! – the geometry of a Graßmann-Cayley algebra, i.e. projective geometry is by the way the first *application* of Graßmann's work by himself [64]. Nowadays these topics can be considered in their relation to Graßmann Hopf gebras. The crucial 'regressive product' of Graßmann can easily be defined, again following Rota et al. [43, 117, 83, 11], by Hopf algebra methods. A different route also following Graßmann's first attempt is discussed in Browne [26]. Rota et al., however, used a Peano space, a pair of a linear space V and a volume to come up with invariant theoretic methods. It turns out, and is in fact implemented in BIGEBRA this way [6, 7], that meet and join operations of projective geometry are encoded most efficiently and mathematically sound using Graßmann Hopf gebra. Graßmannians, flag manifolds which are important in string theory, M-theory, robotics and various other objects from algebraic geometry can be reached in this framework with great formal and computational ease.

It turned out to be extremely useful to have geometrical ideas at hand which can be transformed into the QF theoretical framework. As a general rule, it is true that sane geometric concepts translate into sane concepts of QFT. However a complete treatment of the geometric background would have brought us too far off the road. Examples of such geometries would be Möbius geometry, Laguerre geometry, projective and incidence geometries, Hijelmslev planes and groups etc. [71, 15, 9, 10, 140]. I decided to come up with the algebraic part of Peano space, Graßmann-Cayley algebra, meet and join to have them available for later usage. Nevertheless, it will be possible for the interested reader to figure out to a large extend which geometric operations are behind many QF theoretical operations.

In writing a treatise on QCAs, I assume that the reader is familiar with basic facts about Graßmann and Clifford algebras. Reasonable introductions can be found in various text books, e.g. [115, 112, 14, 18, 27, 40, 87]. A good source is also provided by the conference volumes of the five international Clifford conferences [32, 93, 19, 42, 5, 120]. Nevertheless, the terminology needed later on is provided in the text.

In this treatise we make to a large extend use of graphical calculi. These methods turn out to be efficient, inspiring and allow to memorize particular equations in an elegant way, e.g. the 'Rota-sausage' of cliffordization which is explained in the text. Complicated calculations can be turned into easy manipulations of graphs. This is one key point which is already well established, another issue is to explore the topological and other properties of the involved graphs. This would lead us to graph theory itself, combinatorial topology, but also to the exciting topic of matroid theory. However, we have avoided graph theory, topology and matroids in this work.

Mathematics provides several graphical calculi. We have decided to use three flavours of them. I: Kuperberg's translation of tensor algebra using a self-created very intuitive method because we require some of his important results. Many current papers are based on a couple of lemmas proved in his writings. II. Commutative diagrams constitute a sort of *lingua franca* in mathematics. III. Tangle diagrams turn out to be dual to commutative diagrams in a particular sense. From a physicist's point of view they constitute a much more natural way to display dynamical 'processes'.

Of course, graphical calculi are present in physics too, especially in QFT and for the tensor or spinor algebra, e.g. [106] appendix. The well known Feynman graphs are a particular case of a successful graphical calculus in QFT. Connes-Kreimer renormalization attacks QFT via this route. Following Cayley, rooted trees are taken to encode the complexity of differentiation which leads via the Butcher B-series [28, 29] and a 'decoration' technique to the Zimmermann forest formulas of BPHZ (Bogoliubov-Parasiuk-Hepp-Zimmermann) renormalization in momentum space.

Our work makes contact to QFT on a different and very solid way not using the mathematically peculiar path integral, but functional differential equations of functional quantum field theory, a method developed by Stumpf and coll. [128, 17]. This approach takes its starting point in position space and proceeds by implementing an algebraic framework inspired by and closely related to C^*-algebraic methods without assuming positivity.

However, this method was not widely used in spite of reasonable and unique achievements, most likely due to its lengthy and cumbersome calculations. When I became aware of Clifford algebras in 1993, as promoted by D. Hestenes [68, 69] for some decades now, it turns out that this algebraic structure is a key step to compactify notation and calculations of functional QFT [47]. In the same time many *ad hoc* arguments have been turned into a mathematical sound formulation, see e.g. [47, 48, 60, 50]. But renormalization was still not in the game, mostly since in Stumpf's group in Tübingen the main interest was laid on non-linear spinor field theory which has to be regularized since it is non-renormalizable.

While I was finishing this treatise Christian Brouder came up in January 2002 with an idea how to employ cliffordization in renormalization theory. He used the same transition as was employed in [56] to pass from normal- to time-ordered operator products and correlation functions but implemented an additional bilinear form which introduces the renormalization parameters into the theory but remains in the framework of cliffordization. This is the last part of a puzzle which is needed to formulate all of the algebraic aspects of (perturbative) QFT entirely using the cliffordization technique and therefore in the framework of a Clifford Hopf gebra (Brouder's term is 'quantum field algebra', [22]). This event caused a prolongation by a chapter on generalized cliffordization in the mathematical part in favour of some QFT which was removed and has to be rewritten along entirely hopfish lines. It does not make any sense to go with the *algebra only* description any longer. As a consequence, the discussion of QFT under the topic 'QFT as Clifford Hopf gebra' will be a sort of second volume to this work. Nevertheless, we give a complete synopsis of QFT in terms of QCAs, i.e. in terms of Clifford Hopf gebras. Many results can, however, be found in a pre-Hopf status in our publications.

What is the content and what are the *main results*?

- The Peano space and the Graßmann-Cayley algebra, also called bracket algebra, are treated in its classical form as also in the Hopf algebraic context.

- The bracket of invariant theory is related to a Hopf gebraic integral.

- Five methods are exhibited to construct (quantum) Clifford algebras, showing the outstanding beautiness of the Hopf gebraic method of cliffordization.

- We give a detailed account on Quantum Clifford Algebras (QCA) based on an arbitrary bilinear form B having no particular symmetry.

- We compare Hopf *algebras* and Hopf *gebras*, the latter providing a much more plain development of the theory.

- Following Oziewicz, we present Hopf gebra theory. The crossing and the antipode are exhibited as dependent structures which have to be calculated from structure tensors of the product and co-product of a bi-convolution and cannot be subjected to a choice.

- We use Hopf algebraic methods to derive the basic formulas of Clifford algebra theory (classical and QCA). One of them will be called *Pieri-formula of Clifford algebra*.
- We discuss the Rota-Stein cliffordization and co-cliffordization, which will be called, stressing an analogy, the *Littlewood-Richardson rule of Clifford algebra*.
- We derive *grade free* and very efficient product formulas for almost all products of Clifford and Graßmann-Cayley algebras, e.g. Clifford product, Clifford co-product (time- and normal-ordered operator products and correlation functions based on dotted and undotted exterior wedge products), meet and join products, co-meet and co-join, left and right contraction by arbitrary elements, left and right co-contractions, etc.
- We introduce non-interacting and interacting Hopf gebras which cures a drawback in an important lemma of Kuperberg which is frequently used in the theory of integrable systems, knots and even QFT as proposed by Witten. Their setting turns thereby out to be close to free theories.
- We show in low dimensional examples that no non-trivial integrals do exist in Clifford co-gebras and conjecture this to be generally true.
- A 'spinorial' antipode, a convolutive unipotent, is given which symmetrizes the Kuperberg ladder.
- We extend cliffordization to bilinear forms $\mathcal{BF}$ which are *not* derivable from the exponentiation of a bilinear form on the generating space B.
- We discuss generalized cliffordization based on non-exponentially generated bilinear forms. Assertions on the derived product show that exponentially generated bilinear forms are related to 2-cocycles.
- An overview is presented on functional QFT. Generating functionals are derived for time- and normal-ordered non-linear spinor field theory and spinor electrodynamics.
- A detailed account on the role of the counit as a 'vacuum' state is described. Two models with $U(1)$ and $U(2)$ symmetry are taken as examples.
- It is shown how the quantization enters the cliffordization. Furthermore we explain in which way the vacuum is determined by the propagator of the theory.
- Quantum Clifford algebras are proposed as the algebras of QFT.

What is *not to be found* in this treatise? It was not intended to develop Clifford algebra theory from scratch, but to concentrate on the 'quantum' part of this structure including the unavoidable hopfish methods. q-deformation, while possible and most likely natural in our framework is not explicitely addressed. However the reader should consult our results presented in Refs.

[51, 54, 5, 53] where this topic is addressed. A detailed explanation why 'quantum' has been used as prefix in QCA can be found in [57]. Geometry is reduced to algebra, which is a pity. A broader treatment, e.g. Clifford algebras over finite fields, higher geometries, incidence geometries, Hjielmslev planes etc. was not fitting coherently into this work and would have fatten it becoming thereby unhandsome. An algebro-synthetic approach to geometry would also constitute another volume which would be worth to be written. This is not a work in mathematics, especially not a sort of 'Bourbaki chapter' where a mathematical field is developed straightforward to its highest extend providing all relevant definitions and proving all important theorems. We had to concentrate on hot spots for lack of time and space and to come to a status where the method can be applied and prove its value. The symmetric group algebra and its deformation, the Hecke algebra, had to be postponed, as also a discussion of Young tableaux and their relation to Specht modules and Schubert varieties. And many more exciting topics . . .

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Konstanz, January 25, 2002
Bertfried Fauser

Wir armen Menschenkinder
sind eitel arme Sünder
und wissen garnicht viel
wir spinnen Luftgespinste
und suchen viele Künste
und kommen weiter von dem Ziel!

Matthias Claudius