Preface

Universal algebra has enjoyed a particularly explosive growth in the last twenty years, and a student entering the subject now will find a bewildering amount of material to digest.

This text is not intended to be encyclopedic; rather, a few themes central to universal algebra have been developed sufficiently to bring the reader to the brink of current research. The choice of topics most certainly reflects the authors' interests.

Chapter I contains a brief but substantial introduction to lattices, and to the close connection between complete lattices and closure operators. In particular, everything necessary for the subsequent study of congruence lattices is included.

Chapter II develops the most general and fundamental notions of universal algebra—these include the results that apply to all types of algebras, such as the homomorphism and isomorphism theorems. Free algebras are discussed in great detail—we use them to derive the existence of simple algebras, the rules of equational logic, and the important Mal'cev conditions. We introduce the notion of classifying a variety by properties of (the lattices of) congruences on members of the variety. Also, the center of an algebra is defined and used to characterize modules (up to polynomial equivalence).

In Chapter III we show how neatly two famous results—the refutation of Euler's conjecture on orthogonal Latin squares and Kleene's characterization of languages accepted by finite automata—can be presented using universal algebra. We predict that such "applied universal algebra" will become much more prominent.

Chapter IV starts with a careful development of Boolean algebras, including Stone duality, which is subsequently used in our study of Boolean sheaf representations; however, the cumbersome formulation of general sheaf theory has been replaced by the considerably simpler definition of a Boolean product. First we look at Boolean powers, a beautiful tool for transferring results about Boolean algebras to other varieties as well as for providing a structure theory for certain varieties. The highlight of the chapter is the study of discriminator varieties. These varieties have played a remarkable role in the study of spectra, model companions, decidability, and Boolean product representations. Probably no other class of varieties is so well-behaved yet so fascinating.

The final chapter gives the reader a leisurely introduction to some basic concepts, tools, and results of model theory. In particular, we use the ultraproduct construction to derive the compactness theorem and to prove fundamental preservation theorems. Principal congruence formulas are a favorite model-theoretic tool of universal algebraists, and we use them in the study of the sizes of subdirectly irreducible algebras. Next we prove three general results on the existence of a finite basis for an equational theory. The last topic is semantic embeddings, a popular technique for proving undecidability results. This technique is essentially algebraic in nature, requiring no familiarity whatsoever with the theory of algorithms. (The study of decidability has given surprisingly deep insight into the limitations of Boolean product representations.)

At the end of several sections the reader will find selected references to source material plus state of the art texts or papers relevant to that section, and at the end of the book one finds a brief survey of recent developments and several outstanding problems.

The material in this book divides naturally into two parts. One part can be described as "what every mathematician (or at least every algebraist) should know about universal algebra." It would form a short introductory course to universal algebra, and would consist of Chapter I; Chapter II except for §4, §12, §13, and the last parts of §11, §14; Chapter IV §1–4; and Chapter V §1 and the part of §2 leading to the compactness theorem. The remaining material is more specialized and more intimately connected with current research in universal algebra.

Chapters are numbered in Roman numerals I through V, the sections in a chapter are given by Arabic numerals, §1, §2, etc. Thus II§6.18 refers to item 18, which happens to be a theorem, in Section 6 of Chapter II. A citation within Chapter II would simply refer to this item as 6.18. For the exercises we use numbering such as II§5 Exercise 4, meaning the fourth exercise in §5 of Chapter II. The bibliography is divided into two parts, the first containing books and survey articles, and the second research papers. The books and survey articles are referred to by number, e.g., G. Birkhoff [3], and the research papers by year, e.g., R. McKenzie [1978].