Preface

The last treatise on the theory of determinants, by T. Muir, revised and enlarged by W.H. Metzler, was published by Dover Publications Inc. in 1960. It is an unabridged and corrected republication of the edition originally published by Longman, Green and Co. in 1933 and contains a preface by Metzler dated 1928. The Table of Contents of this treatise is given in Appendix 13.

A small number of other books devoted entirely to determinants have been published in English, but they contain little if anything of importance that was not known to Muir and Metzler. A few have appeared in German and Japanese. In contrast, the shelves of every mathematics library groan under the weight of books on linear algebra, some of which contain short chapters on determinants but usually only on those aspects of the subject which are applicable to the chapters on matrices. There appears to be tacit agreement among authorities on linear algebra that determinant theory is important only as a branch of matrix theory. In sections devoted entirely to the establishment of a determinantal relation, many authors define a determinant by first defining a matrix $\mathbf{M}$ and then adding the words: "Let $\det \mathbf{M}$ be the determinant of the matrix $\mathbf{M}$" as though determinants have no separate existence. This belief has no basis in history. The origins of determinants can be traced back to Leibniz (1646–1716) and their properties were developed by Vandermonde (1735–1796), Laplace (1749–1827), Cauchy (1789–1857) and Jacobi (1804–1851) whereas matrices were not introduced until the year of Cauchy's death, by Cayley (1821–1895). In this book, most determinants are defined directly.

It may well be perfectly legitimate to regard determinant theory as a branch of matrix theory, but it is such a large branch and has such large and independent roots, like a branch of a banyan tree, that it is capable of leading an independent life. Chemistry is a branch of physics, but it is sufficiently extensive and profound to deserve its traditional role as an independent subject. Similarly, the theory of determinants is sufficiently extensive and profound to justify independent study and an independent book.

This book contains a number of features which cannot be found in any other book. Prominent among these are the extensive applications of scaled cofactors and column vectors and the inclusion of a large number of relations containing derivatives. Older books give their readers the impression that the theory of determinants is almost entirely algebraic in nature. If the elements in an arbitrary determinant A are functions of a continuous variable x, then A possesses a derivative with respect to x. The formula for this derivative has been known for generations, but its application to the solution of nonlinear differential equations is a recent development.

The first five chapters are purely mathematical in nature and contain old and new proofs of several old theorems together with a number of theorems, identities, and conjectures which have not hitherto been published. Some theorems, both old and new, have been given two independent proofs on the assumption that the reader will find the methods as interesting and important as the results.

Chapter 6 is devoted to the applications of determinants in mathematical physics and is a unique feature in a book for the simple reason that these applications were almost unknown before 1970, only slowly became known during the following few years, and did not become widely known until about 1980. They naturally first appeared in journals on mathematical physics of which the most outstanding from the determinantal point of view is the *Journal of the Physical Society of Japan*. A rapid scan of Section 15A15 in the *Index of Mathematical Reviews* will reveal that most pure mathematicians appear to be unaware of or uninterested in the outstanding contributions to the theory and application of determinants made in the course of research into problems in mathematical physics. These usually appear in Section 35Q of the Index. Pure mathematicians are strongly recommended to make themselves acquainted with these applications, for they will undoubtedly gain inspiration from them. They will find plenty of scope for purely analytical research and may well be able to refine the techniques employed by mathematical physicists, prove a number of conjectures, and advance the subject still further. Further comments on these applications can be found in the introduction to Chapter 6.

There appears to be no general agreement on notation among writers on determinants. We use the notion $A_n = |a_{ij}|_n$ and $B_n = |b_{ij}|_n$, where i and j are row and column parameters, respectively. The suffix n denotes the order of the determinant and is usually reserved for that purpose. Rejecter minors of A_n are denoted by $M_{ij}^{(n)}$, etc., retainer minors are denoted by N_{ij}, etc., simple cofactors are denoted by $A_{ij}^{(n)}$, etc., and scaled cofactors are denoted by A_n^{ij}, etc. The n may be omitted from any passage if all the determinants which appear in it have the same order. The letter D, sometimes with a suffix x, t, etc., is reserved for use as a differential operator. The letters h, i, j, k, m, p, q, r, and s are usually used as integer parameters. The letter l is not used in order to avoid confusion with the unit integer. Complex numbers appear in some sections and pose the problem of conflicting priorities. The notation $\omega^2 = -1$ has been adopted since the letters i and j are indispensable as row and column parameters, respectively, in passages where a large number of such parameters are required. Matrices are seldom required, but where they are indispensable, they appear in boldface symbols such as **A** and **B** with the simple convention $A = \det \mathbf{A}$, $B = \det \mathbf{B}$, etc. The boldface symbols **R** and **C**, with suffixes, are reserved for use as row and column vectors, respectively. Determinants, their elements, their rejecter and retainer minors, their simple and scaled cofactors, their row and column vectors, and their derivatives have all been expressed in a notation which we believe is simple and clear and we wish to see this notation adopted universally.

The Appendix consists mainly of nondeterminantal relations which have been removed from the main text to allow the analysis to proceed without interruption.

The Bibliography contains references not only to all the authors mentioned in the text but also to many other contributors to the theory of determinants and related subjects. The authors have been arranged in alphabetical order and reference to *Mathematical Reviews*, *Zentralblatt für Mathematik*, and *Physics Abstracts* have been included to enable the reader who has no easy access to journals and books to obtain more details of their contents than is suggested by their brief titles.

The true title of this book is *The Analytic Theory of Determinants with Applications to the Solutions of Certain Nonlinear Equations of Mathematical Physics*, which satisfies the requirements of accuracy but lacks the virtue of brevity. Chapter 1 begins with a brief note on Grassmann algebra and then proceeds to define a determinant by means of a Grassmann identity. Later, the Laplace expansion and a few other relations are established by Grassmann methods. However, for those readers who find this form of algebra too abstract for their tastes or training, classical proofs are also given. Most of the contents of this book can be described as complicated applications of classical algebra and differentiation.

In a book containing so many symbols, misprints are inevitable, but we hope they are obvious and will not obstruct our readers' progress for long. All reports of errors will be warmly appreciated.

We are indebted to our colleague, Dr. Barry Martin, for general advice on computers and for invaluable assistance in algebraic computing with the

Maple system on a Macintosh computer, especially in the expansion and factorization of determinants. We are also indebted by Lynn Burton for the most excellent construction and typing of a complicated manuscript in Microsoft Word programming language Formula on a Macintosh computer in camera-ready form.

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