Contents

Preface **v**

1 Determinants, First Minors, and Cofactors **1**
1.1 Grassmann Exterior Algebra 1
1.2 Determinants 1
1.3 First Minors and Cofactors 3
1.4 The Product of Two Determinants — 1 5

2 A Summary of Basic Determinant Theory **7**
2.1 Introduction 7
2.2 Row and Column Vectors 7
2.3 Elementary Formulas 8
2.3.1 Basic Properties 8
2.3.2 Matrix-Type Products Related to Row and Column Operations 10
2.3.3 First Minors and Cofactors; Row and Column Expansions 12
2.3.4 Alien Cofactors; The Sum Formula 12
2.3.5 Cramer's Formula 13
2.3.6 The Cofactors of a Zero Determinant 15
2.3.7 The Derivative of a Determinant 15

3 Intermediate Determinant Theory **16**
3.1 Cyclic Dislocations and Generalizations 16
3.2 Second and Higher Minors and Cofactors 18

3.2.1 Rejecter and Retainer Minors 18
3.2.2 Second and Higher Cofactors 19
3.2.3 The Expansion of Cofactors in Terms of Higher Cofactors 20
3.2.4 Alien Second and Higher Cofactors; Sum Formulas 22
3.2.5 Scaled Cofactors 23
3.3 The Laplace Expansion 25
3.3.1 A Grassmann Proof 25
3.3.2 A Classical Proof 27
3.3.3 Determinants Containing Blocks of Zero Elements . 30
3.3.4 The Laplace Sum Formula 32
3.3.5 The Product of Two Determinants — 2 33
3.4 Double-Sum Relations for Scaled Cofactors 34
3.5 The Adjoint Determinant 36
3.5.1 Definition 36
3.5.2 The Cauchy Identity 36
3.5.3 An Identity Involving a Hybrid Determinant . . . 37
3.6 The Jacobi Identity and Variants 38
3.6.1 The Jacobi Identity — 1 38
3.6.2 The Jacobi Identity — 2 41
3.6.3 Variants 43
3.7 Bordered Determinants 46
3.7.1 Basic Formulas; The Cauchy Expansion 46
3.7.2 A Determinant with Double Borders 49

4 Particular Determinants 51
4.1 Alternants 51
4.1.1 Introduction 51
4.1.2 Vandermondians 52
4.1.3 Cofactors of the Vandermondian 54
4.1.4 A Hybrid Determinant 55
4.1.5 The Cauchy Double Alternant 57
4.1.6 A Determinant Related to a Vandermondian . . . 59
4.1.7 A Generalized Vandermondian 60
4.1.8 Simple Vandermondian Identities 60
4.1.9 Further Vandermondian Identities 63
4.2 Symmetric Determinants 64
4.3 Skew-Symmetric Determinants 65
4.3.1 Introduction 65
4.3.2 Preparatory Lemmas 69
4.3.3 Pfaffians 73
4.4 Circulants 79
4.4.1 Definition and Notation 79
4.4.2 Factors 79

4.4.3 The Generalized Hyperbolic Functions 81
4.5 Centrosymmetric Determinants 85
4.5.1 Definition and Factorization 85
4.5.2 Symmetric Toeplitz Determinants 87
4.5.3 Skew-Centrosymmetric Determinants 90
4.6 Hessenbergians 90
4.6.1 Definition and Recurrence Relation 90
4.6.2 A Reciprocal Power Series 92
4.6.3 A Hessenberg–Appell Characteristic Polynomial . 94
4.7 Wronskians 97
4.7.1 Introduction 97
4.7.2 The Derivatives of a Wronskian 99
4.7.3 The Derivative of a Cofactor 100
4.7.4 An Arbitrary Determinant 102
4.7.5 Adjunct Functions 102
4.7.6 Two-Way Wronskians 103
4.8 Hankelians 1 104
4.8.1 Definition and the ϕ_m Notation 104
4.8.2 Hankelians Whose Elements are Differences 106
4.8.3 Two Kinds of Homogeneity 108
4.8.4 The Sum Formula 108
4.8.5 Turanians 109
4.8.6 Partial Derivatives with Respect to ϕ_m 111
4.8.7 Double-Sum Relations 112
4.9 Hankelians 2 115
4.9.1 The Derivatives of Hankelians with Appell Elements 115
4.9.2 The Derivatives of Turanians with Appell and Other Elements 119
4.9.3 Determinants with Simple Derivatives of All Orders 122
4.10 Henkelians 3 123
4.10.1 The Generalized Hilbert Determinant 123
4.10.2 Three Formulas of the Rodrigues Type 127
4.10.3 Bordered Yamazaki–Hori Determinants — 1 129
4.10.4 A Particular Case of the Yamazaki–Hori Determinant 135
4.11 Hankelians 4 137
4.11.1 v-Numbers 137
4.11.2 Some Determinants with Determinantal Factors . 138
4.11.3 Some Determinants with Binomial and Factorial Elements 142
4.11.4 A Nonlinear Differential Equation 147
4.12 Hankelians 5 153
4.12.1 Orthogonal Polynomials 153

4.12.2 The Generalized Geometric Series and Eulerian Polynomials . . . 157
4.12.3 A Further Generalization of the Geometric Series . 162
4.13 Hankelians 6 . . . 165
4.13.1 Two Matrix Identities and Their Corollaries . . . 165
4.13.2 The Factors of a Particular Symmetric Toeplitz Determinant . . . 168
4.14 Casoratians — A Brief Note . . . 169

5 Further Determinant Theory 170
5.1 Determinants Which Represent Particular Polynomials . . 170
5.1.1 Appell Polynomial . . . 170
5.1.2 The Generalized Geometric Series and Eulerian Polynomials . . . 172
5.1.3 Orthogonal Polynomials . . . 174
5.2 The Generalized Cusick Identities . . . 178
5.2.1 Three Determinants . . . 178
5.2.2 Four Lemmas . . . 180
5.2.3 Proof of the Principal Theorem . . . 183
5.2.4 Three Further Theorems . . . 184
5.3 The Matsuno Identities . . . 187
5.3.1 A General Identity . . . 187
5.3.2 Particular Identities . . . 189
5.4 The Cofactors of the Matsuno Determinant . . . 192
5.4.1 Introduction . . . 192
5.4.2 First Cofactors . . . 193
5.4.3 First and Second Cofactors . . . 194
5.4.4 Third and Fourth Cofactors . . . 195
5.4.5 Three Further Identities . . . 198
5.5 Determinants Associated with a Continued Fraction . . . 201
5.5.1 Continuants and the Recurrence Relation . . . 201
5.5.2 Polynomials and Power Series . . . 203
5.5.3 Further Determinantal Formulas . . . 209
5.6 Distinct Matrices with Nondistinct Determinants . . . 211
5.6.1 Introduction . . . 211
5.6.2 Determinants with Binomial Elements . . . 212
5.6.3 Determinants with Stirling Elements . . . 217
5.7 The One-Variable Hirota Operator . . . 221
5.7.1 Definition and Taylor Relations . . . 221
5.7.2 A Determinantal Identity . . . 222
5.8 Some Applications of Algebraic Computing . . . 226
5.8.1 Introduction . . . 226
5.8.2 Hankel Determinants with Hessenberg Elements . 227
5.8.3 Hankel Determinants with Hankel Elements . . . 229

5.8.4 Hankel Determinants with Symmetric Toeplitz Elements . . . 231
5.8.5 Hessenberg Determinants with Prime Elements . . 232
5.8.6 Bordered Yamazaki–Hori Determinants — 2 232
5.8.7 Determinantal Identities Related to Matrix Identities . . . 233

6 Applications of Determinants in Mathematical Physics 235
6.1 Introduction . . . 235
6.2 Brief Historical Notes . . . 236
6.2.1 The Dale Equation . . . 236
6.2.2 The Kay–Moses Equation . . . 237
6.2.3 The Toda Equations . . . 237
6.2.4 The Matsukidaira–Satsuma Equations . . . 239
6.2.5 The Korteweg–de Vries Equation . . . 239
6.2.6 The Kadomtsev–Petviashvili Equation . . . 240
6.2.7 The Benjamin–Ono Equation . . . 241
6.2.8 The Einstein and Ernst Equations . . . 241
6.2.9 The Relativistic Toda Equation . . . 245
6.3 The Dale Equation . . . 246
6.4 The Kay–Moses Equation . . . 249
6.5 The Toda Equations . . . 252
6.5.1 The First-Order Toda Equation . . . 252
6.5.2 The Second-Order Toda Equations . . . 254
6.5.3 The Milne-Thomson Equation . . . 256
6.6 The Matsukidaira–Satsuma Equations . . . 258
6.6.1 A System With One Continuous and One Discrete Variable . . . 258
6.6.2 A System With Two Continuous and Two Discrete Variables . . . 261
6.7 The Korteweg–de Vries Equation . . . 263
6.7.1 Introduction . . . 263
6.7.2 The First Form of Solution . . . 264
6.7.3 The First Form of Solution, Second Proof . . . 268
6.7.4 The Wronskian Solution . . . 271
6.7.5 Direct Verification of the Wronskian Solution . . . 273
6.8 The Kadomtsev–Petviashvili Equation . . . 277
6.8.1 The Non-Wronskian Solution . . . 277
6.8.2 The Wronskian Solution . . . 280
6.9 The Benjamin–Ono Equation . . . 281
6.9.1 Introduction . . . 281
6.9.2 Three Determinants . . . 282
6.9.3 Proof of the Main Theorem . . . 285
6.10 The Einstein and Ernst Equations . . . 287
6.10.1 Introduction . . . 287

6.10.2 Preparatory Lemmas 287
6.10.3 The Intermediate Solutions 292
6.10.4 Preparatory Theorems 295
6.10.5 Physically Significant Solutions 299
6.10.6 The Ernst Equation 302
6.11 The Relativistic Toda Equation — A Brief Note 302

A **304**
A.1 Miscellaneous Functions 304
A.2 Permutations 307
A.3 Multiple-Sum Identities 311
A.4 Appell Polynomials 314
A.5 Orthogonal Polynomials 321
A.6 The Generalized Geometric Series and Eulerian Polynomials 323
A.7 Symmetric Polynomials 326
A.8 Differences 328
A.9 The Euler and Modified Euler Theorems on Homogeneous Functions 330
A.10 Formulas Related to the Function $(x + \sqrt{1 + x^2})^{2n}$ 332
A.11 Solutions of a Pair of Coupled Equations 335
A.12 Bäcklund Transformations 337
A.13 Muir and Metzler, A Treatise on the Theory of Determinants 341

Bibliography **343**

Index **373**